Layered Graphs with a Maximum Number of Edges

E. Fiesler[a]

[a]IDIAP, c.p. 609, CH-1920 Martigny, Switzerland

Layered graphs are introduced, together with their characteristic edges (connections) and their accompanying interconnection schemes. For each of the possible interconnection schemes the topology with the maximum number of edges is analytically computed and the results presented in a theorem[1].

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1. Introduction

Define a *layer* as an unordered set of vertices v_i and let $|v_i| = N_l$, the *size* of the layer, and l the ordinal number of the layer. A *framework* can then be defined as an ordered set of layers: $F = \{v_{i_1}, v_{i_2}, \ldots, v_{i_L}\}$, where L is the total number of layers. A *layered graph* is then a tuple (F, E), where E is the set of edges[2] interconnecting the vertices.

If N is the total number of vertices in the graph,

$$N = \sum_{l=1}^{L} N_l. \tag{1}$$

Given the layered structure, one can discern three kinds of edges:

Definition : An *interlayer connection* is a connection between vertices in adjacent layers of the graph.

Definition : An *intralayer connection* is a connection between vertices of the same layer in the graph.

A sub-class of intralayer connections are self-connections:

Definition : A *self-connection* is a connection which originates and terminates at the same vertex.

Definition : A *supralayer connection* is a connection between vertices that are neither in adjacent layers, nor in the same layer of the graph, that is, they 'skip' at least one layer.

Based on these three kinds of edges, the following *interconnection schemes* can be defined:

- *interlayer connected*
- *interlayer and intralayer connected*

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[1]Due to space limitations, the proofs of theorems two and three are omitted. They can be found in a forthcoming comprehensive publication: [1].

[2]In this publication, the words 'edge' and 'connection' are used interchangeably.

- *interlayer and supralayer connected*
- allowing all three kinds of connections, which is equivalent to a complete graph.

In order to calculate the maximum number of connections for each of these interconnection schemes, the total number of connections, given a fixed framework, has to be determined for them.

Layered graphs can represent the underlying structure of networks in diverse disciplines, like computer (interconnection) networking: *multistage networks* [2] [3] and *banyan networks* [4], (switching) circuit analysis: *cascaded networks* ([5]), and other network based fields. The theory presented in this document has been applied to neural networks [6].

2. Counting Edges

2.1. Interlayer connections

The number of connections between two adjacent layers in a *fully interlayer connected* graph is equal to the product of the number of vertices in each of the layers. The total number of connections for the graph (C) is obtained by a summation over the layers:

$$C = \sum_{l=2}^{L} N_{l-1}\, N_l. \tag{2}$$

2.2. Interlayer and intralayer connections

In the case that the graph has both interlayer and intralayer connections, a number equal to the number of possible connections within a layer ($\frac{N_l}{2}\left(N_l \pm 1\right)$) has to be added to equation (2) for each layer in the graph. The $\pm$-symbol denotes the option of having self-connections. If self-connections are present, addition has to be used, and subtraction otherwise. The total number of connections becomes therefore

$$\sum_{l=1}^{L} \frac{N_l}{2}\left(N_l \pm 1\right) + \sum_{l=2}^{L} N_{l-1}\, N_l = \frac{N_1^2 \pm N}{2} + \sum_{l=2}^{L} N_l \left(N_{l-1} + \frac{N_l}{2}\right). \tag{3}$$

2.3. Interlayer and supralayer connections

The number of connections in a layered graph which, besides all interlayer connections, also has all supralayer connections, can be calculated by summing over all the vertices in all the layers, multiplied by the number of vertices in all the layers of a higher index.

$$C = \sum_{l=2}^{L} N_{l-1} \sum_{m=l}^{L} N_m = \sum_{m=1}^{L-1} N_{m+1} \sum_{l=1}^{m} N_l. \tag{4}$$

2.4. Complete graphs

When combining equations (3) and (4), the number of connections for a complete undirected graph is obtained:

$$C = \sum_{l=1}^{L} \frac{N_l}{2}\left(N_l \pm 1\right) + \sum_{l=2}^{L} N_{l-1} \sum_{m=l}^{L} N_m = \frac{N}{2}\left(N \pm 1\right). \tag{5}$$

3. Maximal Interconnection topologies

Depending on the types of interconnections present, the number of connections in a graph can be maximized by varying the topology of the graph. This section (section 3) shows which topologies provide a maximum number of connections for each of the possible interconnection schemes.

3.1. Interlayer connections

For *fully interlayer connected* graphs, the number of connections (C) is $\sum_{l=2}^{L} N_{l-1} N_l$, (equation (2)). First, consider a two layer graph $L=2$, $C=N_1N_2$, and $N = N_1+N_2$. The total number of connections can be represented as a function of N_1 and N_2: $C(N_1, N_2) = N_1N_2$. Using $N = N_1 + N_2$, C can be transformed into a function of N_1 only. $C(N_1) = N_1(N-N_1)$. To maximize $C(N_1)$, the partial derivative of $C(N_1)$ with respect to N_1 has to be determined: $\frac{\partial C(N_1)}{\partial N_1} = N - 2N_1$. By setting this quantity equal to zero, a (global) maximum of

$$\frac{N^2}{4} \text{ is found at } N_1 = N_2 = \frac{N}{2}.$$

Now consider a three layer system: $L=3$ and $C=N_2(N_1+N_3)$, which gives analogously

$$C(N_1, N_2) = C(N_2, N_3) = N_2(N - N_2)$$

or

$$C(N_1, N_3) = N_1(N - N_1 - 2N_3) + N_3(N - N_3).$$

A maximum of $\frac{N^2}{4}$ occurs when $N_2 = \frac{N}{2}$ and $N_3 = \frac{N}{2} - N_1$.

For more than three layers, the outcome of the maximization procedure is: drop all but two or three layers, and put half of the total number of vertices in layer two, and the same maximum will hold. This is formalized by the following theorem:

Theorem 1 *The maximum interconnection topology for a fully interlayer connected graph consists of two or three layers, with* $\left\lfloor \frac{N}{2} \right\rfloor$ *or* $\left\lceil \frac{N}{2} \right\rceil$ *vertices*[3] *in the second layer. The total number of connections is then* $\left\lfloor \frac{N^2}{4} \right\rfloor$.

Proof A proof is given in Appendix A. □

3.2. Interlayer and intralayer connections

For graphs with both interlayer and intralayer connections, the maximal interconnection topologies can be determined analogously. For these graphs, one or two layer systems are optimal. For more than two layers the outcome of the optimization is: drop all but one or two layers and distribute the vertices over these layers. One and two layer graphs which have all possible interlayer and intralayer connections (including self-connections) are equivalent to complete graphs with the same number of vertices. The maximum

[3] $\lfloor x \rfloor$ denotes the 'floor function', which gives the biggest integer smaller than or equal to x, and $\lceil x \rceil$ denotes the 'ceiling function', which gives the smallest integer greater than or equal to x.

number of connections $C_{\max} = \frac{N}{2}(N \pm 1)$ is therefore the same as for the complete graph. So there is no absolute maximum; any distribution of the vertices over the two layers gives this "maximum".

Theorem 2 *The maximal interconnection topology for a fully interlayer and intralayer connected graph is one or two layers with no restriction on the distribution of the vertices over the layers, in the case of two layers, with* $C_{\max} = \frac{N(N\pm1)}{2}$.

3.3. Interlayer and supralayer connections

Since two layer graphs do not have supralayer connections, the smallest graphs with both interlayer and supralayer connections are three layer graphs: $L=3$ and $C=N_1N_2 + N_1N_3 + N_2N_3$. C can be written as a function of two variables again.

$$C(N_1, N_2) = N_1(N - N_1 - N_2) + N_2(N - N_2),$$

$$C(N_1, N_3) = N_3(N - N_1 - N_3) + N_1(N - N_1),$$

and

$$C(N_2, N_3) = N_2(N - N_2 - N_3) + N_3(N - N_3).$$

Maximization gives a maximum at $N_1 = N_2 = N_3 = \frac{N}{3}$. The maximum for C is then $\frac{N^2}{3}$. For more than three layers, the maximum interconnection topology is also: divide the vertices equally over the available layers. This gives

Theorem 3 *The maximum interconnection topology for a fully interlayer and supralayer connected graph is three or more layers with the vertices equally divided over the layers. This gives a total number of connections of* $\frac{N(L-1)(N+2n)+n(n-1)}{2L}$, *where* $n = N - L\left\lfloor\frac{N}{L}\right\rfloor$.

Thus, in the case of both interlayer and supralayer connections, optimization yields: the more layers the better. Note that

$$\lim_{L\to\infty, n\to 0} \frac{N(L-1)(N+2n)+n(n-1)}{2L} = \frac{N^2}{2}. \tag{6}$$

3.4. Complete graphs

Since all complete graphs are topologically identical, the following theorem holds:

Theorem 4 *The topology of all complete graphs is maximally interconnected, and the maximum number of connections is* $\frac{N(N+1)}{2}$.

Proof The proof is trivial, since the topology of a complete graph is equivalent to a fully connected (undirected) graph, whose number of edges only depends on the number of vertices. □

4. Summary and Conclusions

Layered graphs and their accompanying interconnection schemes are introduced. For each of the interconnection schemes, the total number of connections is determined. Further, the maximal interconnection topology for each of them is obtained in the form of a series of analytically proved theorems.

The maximal interconnection topology and maximum number of connections for all the interconnection schemes for fully connected layered graphs (without constraints) are given in the following table, where $L_{\min}$ represents the minimum number of layers, and $L_{\max}$ the maximum number of layers. In this table, where intralayer connections are present, self-connections are assumed to be included, since the presence or absence of self-connections does not alter the optima.

interconnection scheme	*maximal interconnection topology*			
	$L_{\min}$	$L_{\max}$	N_l's	$C_{\max}$
inter	2	3	$N_2 = \left\lfloor \frac{N}{2} \right\rfloor$ or $\left\lceil \frac{N}{2} \right\rceil$	$\left\lfloor \frac{N^2}{4} \right\rfloor$
inter & intra	1	2	any distribution	$\frac{N(N+1)}{2}$
inter & supra	3	no max.	$\forall l : N_l = \left\lfloor \frac{N}{L} \right\rfloor$ or $\left\lceil \frac{N}{L} \right\rceil$	$\left\lfloor \frac{N^2(L-1)}{2L} \right\rfloor$
complete graph	1	no max.	any distribution	$\frac{N(N+1)}{2}$

Appendix A.

Proof of theorem 1 on the maximal interconnection topology for fully interlayer connected graphs.

The problem can be stated as follows: maximize analytically

$$C(N_1, N_2, \cdots, N_L) = \sum_{l=2}^{L} N_{l-1}\, N_l,$$

given the constraints $\sum_{l=1}^{L} N_l = N$, where N is a constant, and $N_l \geq 0, \forall l : 1 < l \leq L$, and $N_1 > 0$.

This is a bounded multivariate non-linear (integer) maximization problem with one linear equality constraint. It can be solved using multidimensional Lagrange multipliers. Define

$$F(N_1, N_2, \cdots, N_L, \lambda) := \sum_{l=2}^{L} N_{l-1}\, N_l + \lambda \left(\sum_{l=1}^{L} N_l - N \right), \tag{7}$$

where λ is the independent multiplier or Lagrange multiplier. The partial derivatives are: $\frac{\partial F}{\partial N_1} = N_2 + \lambda$, $\frac{\partial F}{\partial N_l} = N_{l-1} + N_{l+1} + \lambda$, for $1 < l < L$, $\frac{\partial F}{\partial N_L} = N_{L-1} + \lambda$, and $\frac{\partial F}{\partial \lambda} = \sum_{l=1}^{L} N_l - N$.

Setting them equal to zero gives: $N_2 = -\lambda$, $N_{l-1} = -N_{l+1} - \lambda$, for $1 < l < L$, $N_{L-1} = -\lambda$, and $\sum_{l=1}^{L} N_l = N$.
Mutual substitution gives: $N_l = N_{l+4}$, for $1 \leq l \leq L-4$, $N_2 = -\lambda$, $N_3 = -N_1 - \lambda$, $N_4 = 0$, and $N_{L-1} = -\lambda$.
Four cases can be discerned now, in which x is a non-negative integer.
case 1. $L = 4x + 1$
Solution: $N_l = 0$, $\forall l : 1 \leq l \leq L$, which is discarded, because N_1 has to be positive. (This gives a saddle point.)
case 2. $L = 4x + 2$
Solution: $N_{4k+1} = N_{4k+2} = \frac{2N}{L+2}$ and $N_{4k+3} = N_{4k+4} = 0$, $\forall k : 0 \leq k < x$.
case 3. $L = 4x + 3$
Solution: $N_{4k+2} = \frac{2N}{L+1}$ and $N_{4k+3} = \frac{2N}{L+1} - N_{4k+1}$, $N_{4k+4} = 0$, $\forall k : 0 \leq k < x$.
case 4. $L = 4x + 4$
Solution: $N_{4k+3} = N_{4k+2} = \frac{2N}{L}$ and $N_{4k+4} = N_{4k+1} = 0$, which is also discarded, because here again $N_1 = 0$.

A closer look at cases 2 and 3 reveals that the solutions depend on L and can be further optimized by maximizing them with respect to L. This gives the combined result

$$C_{\max} = \frac{N^2}{4} \quad \text{when} \quad N_1 + N_3 = N_2 = \frac{N}{2}.$$

Note that N_3 can be zero, which will give a two layer solution.

The number of vertices and the number of connections have to be integral numbers. To establish this, N_2 can be chosen $\left\lfloor \frac{N}{2} \right\rfloor$ or $\left\lceil \frac{N}{2} \right\rceil$. The actual number of connections becomes $\left\lfloor \frac{N^2}{4} \right\rfloor$, since the difference $\left(\frac{N^2}{4} - \left\lfloor \frac{N^2}{4} \right\rfloor \right)$ never exceeds $\frac{1}{4}$. □

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