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# Maximum capacity topologies for fully connected layered neural networks with bidirectional connections†

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**Abstract**

One of the main problems in current (artificial) neural network engineering is the lack of design rules for neural networks, i.e. how many layers and how many neurons per layer to choose for a fully connected layered neural network with bidirectional weights. A theory is developed which optimizes the topology of the neural network to allow a maximum potential storage capacity with a minimum amount of neurons.

**Keywords**: (artificial) neural networks, connectionism, neural network topology, neural network statics, neural network connectivity, neural network capacity

## Introduction

Although the field of artificial neural networks, hereafter called neural networks, is a rapidly growing one, some basic questions remain unanswered. One of the most important problems is how to configure a neural network. Many neural network learning rules apply to (fully connected) layered (first order) neural networks with bidirectional weights (or interconnection strengths). A *bidirectional connection* is a connection that has the same connection strength when used for either forward or backward propagation. (If a neural network uses only unidirectional propagation, the interconnection topology of the neural network is identical to one with unidirectional connections.)

For layered neural networks in general, one needs to determine the number of layers and the number of neurons per layer. Since neural networks are used for processing and storage of information, the 'optimal' topology for a neural network is usually one which allows an optimal (information) storage capacity. Since the interconnection strengths (weights) contain the information of the neural network, the information capacity is proportional to the total number of weights in the network [Fiesler-90]. A fully connected neural network will therefore have a higher information capacity than any other interconnection scheme. However, in layered neural networks there are several types of connections.

## Counting Weights

In layered neural networks one can discriminate three classes of connections:
Definition : An *interlayer connection* is a connection between neurons in adjacent layers of the neural network.

Definition : An *intralayer connection* is a connection between neurons of the same layer of the neural network.
Definition : A *supralayer connection* is a connection between neurons that are neither in adjacent layers, nor in the same layer of the neural network.

A sub-class of intralayer connections are self-connections:
Definition : A *self-connection* is a connection which originates and terminates at the same neuron.

A neural network can have all possible connections:
Definition : A *plenary neural network* is a neural network which has all possible interlayer, intralayer, and supralayer connections; in other words it is a 'truly' fully connected neural network.

The total number of weights ($W$) for a neural network with $L$ layers, which has only interlayer connections (i.e. they have neither intralayer nor supralayer connections) is the sum of all possible connections (between each pair of adjacent layers) in the network. The number of connections between two adjacent layers in a fully *inter*connected network is equal to the product of the number of neurons in each of the layers. In order to get the total number of weights for the complete network, a summation is needed over the layers:

$$W = \sum_{l=2}^{L} W_l = \sum_{l=2}^{L} N_{l-1}\, N_l,$$

where $W_l$ stands for the number of weights between layer $l-1$ and $l$, and $N_l$ is the number of neurons in layer $l$. Layer 1 is the input layer and $N_1$ the number of input neurons.

In the case that the neural network has both interlayer and intralayer connections, a number equal to the number of possible connections within a layer has to be added for each layer. The total number of connections becomes thus:

$$\sum_{l=2}^{L} N_{l-1} N_l + \left[\frac{N_1}{2}\,(N_1 \pm 1) +\right] \sum_{l=2}^{L} \frac{N_l}{2}\,(N_l \pm 1) =$$

$$\left[\frac{N_1}{2}\,(N_1 \pm 1) +\right] \sum_{l=2}^{L} N_l \left(N_{l-1} + \frac{N_l \pm 1}{2}\right),$$

where the part between square brackets is optional; it is deleted when no intralayer connections are present in the input layer. The $\pm$-symbol denotes the option for having self-connections. If self-connections are present, addition has to be used, and substraction otherwise.

The number of weights in a network which, besides interlayer connections also has supralayer connections can be calculated by summing over all the neurons in all the layers, multiplied by the number of neurons in all the layers of a higher index:

$$\sum_{l=2}^{L} N_{l-1} \sum_{m=l}^{L} N_m = \sum_{m=1}^{L-1} \sum_{l=1}^{m} N_l N_{m+1}\,.$$

When combining the previous two formulas, the total number of neurons is obtained for a plenary neural network, which has all three types of connections:

$$W = \left[\frac{N_1}{2}\,(N_1 \pm 1) +\right] \sum_{l=2}^{L} \left(\frac{N_l}{2}\,(N_l \pm 1) + N_{l-1} \sum_{m=l}^{L} N_m\right)$$

where the part in square brackets is again optional and used when intralayer connections are desired in the input layer. In the case that the intralayer connections in the input layer are also present, the formula becomes equal to

$$\sum_{l=1}^{L} \frac{N_l}{2}\,(N_l \pm 1) + \sum_{l=2}^{L} N_{l-1} \sum_{m=l}^{L} N_m\,.$$

Since a plenary network can be represented as a fully connected graph, the previous equation is equal to:

$$W = \frac{N}{2}(N \pm 1),$$

(this is the number of edges in a fully connected undirected graph with $N$ vertices); where $N$ is the total number of neurons in the network: $N = \sum_{l=1}^{L} N_l$.

**Optimal topologies**

Depending on the type(s) of interconnections present, the capacity of a neural network can be optimized by varying the topology of the network. Plenary neural networks are a trivial case; there topology is always optimal, since they can be seen as a fully connected (undirected) graph, whose number of edges only depends on the number of vertices.

For layered neural networks with only interlayer connections (the most used topologies), the configuration topology does make a difference. Let the total number of weights $W = \sum_{l=2}^{L} N_{l-1} N_l$, as defined before. For a two layer neural network $L=2$, $W=N_1N_2$, and $N = N_1+N_2$. The total number of weights, $W$ can be represented as a function of $N$: $W(N) = N_1N_2$. Using $N = N_1 + N_2$, $W$ can be transformed into a function of $N_1$: $W(N_1) = N_1(N-N_1)$. To find the optimal topology, the derivative of $W(N_1)$ with respect to $N_1$ has to be determined: $\frac{dW(N_1)}{dN_1} = N - 2N_1$. A maximum is found and this gives the optimum:

$$W = \frac{N^2}{4} \text{ at } N_1 = N_2 = \frac{N}{2}.$$

Since the number of neurons and the number of weights are integral numbers, $W = \left\lfloor \frac{N^2}{4} \right\rfloor$ and $N_2$ can be choosen $\left\lfloor \frac{N}{2} \right\rfloor$ or $\left\lceil \frac{N}{2} \right\rceil$.

The three layer system $L=3$ and $W=N_2(N_1 + N_3)$ gives analogously: $W(N_1, N_2) = W(N_2, N_3) = N_2(N - N_2)$ or $W(N_1, N_3) = N_1(N - N_1 - 2N_3) + N_3(N - N_3)$. Maximization gives a maximum at $N_2 = \frac{N}{2}$ and $N_3 = \frac{N}{2} - N_1$. The maximum for $W$ is again $\left\lfloor \frac{N^2}{4} \right\rfloor$.

For more than three layers, the outcome of the maximization procedure is: drop all but two or three layers, and the same maximum holds; in other words multi-layer systems ($>3$) layers are not optimal. This outcome coincides with the neural network interpretation of Kolmogorov's theorem, which states that the capabilities of a neural network with more than three layers does not exceed the capabilities of a three layer neural network with $2N_1 + 1$ neurons in the hidden layer and only interlayer connections [Hecht-Nielsen-87].

For neural networks with interlayer plus intralayer connections, a fully connected two layer network is equal to a two layer plenary neural network. It has $W = \frac{(N_1+N_2)(N_1+N_2+1)}{2} = \frac{N}{2}(N+1)$ weights. So there is no absolute maximum; any distribution of the neurons over the two layers gives this "maximum". For more than two layers the outcome of the optimization is: drop all but one or two layers and distribute the neurons over these layers. The maximum $W$ is therefore the same as for the plenary neural network.

Layered neural networks with interlayer and supralayer connections have a different optimum: Since two layer neural networks do not have supralayer connections, the smallest networks to study here are three layer networks: $L = 3$ and $W = N_1N_2 + N_1N_3 + N_2N_3$. $W$ can be written as a function of two variables again: $W(N_1, N_2) = N_1(N - N_1 - N_2) + N_2(N - N_2)$, $W(N_1, N_3) = N_3(N - N_1 - N_3) + N_1(N - N_1)$, or $W(N_2, N_3) = N_2(N - N_2 - N_3) + N_3(N - N_3)$. Maximization gives a maximum at $N_1 = N_2 = N_3 = \frac{N}{3}$. The maximum for $W$ is $\frac{N^2}{3}$. This can be generalized and proven for any number of layers. The maximum is found at $N_l = \frac{N}{L}$, for $L > 2$ and $1 \leq l \leq L$, and the maximum is

$$W = \frac{N^2(L-1)}{2L}.$$

Thus in the case of both interlayer and superlayer connections: Since the number of neurons is a positive integer, each layer gets at least $\lfloor \frac{N}{L} \rfloor$ neurons, and the rest of the neurons ($N - L \lfloor \frac{N}{L} \rfloor$) can be distributed over the layers. The number of weights is also a positive integer. The floor function can only be applied for neural networks with less than eight layers, since the maximum deviation between the optimal and the actual number of weights can be as large as $\frac{L}{8}$ 'weights'.

**Conclusions**

The optimal topology and maximum number of connections for all the interconnection schemes are given in this table:

| *interconnection structure* | *optimal topology* | | | $W_{max}$ |
|---|---|---|---|---|
| | $L_{min}$ | $L_{max}$ | $N_l$'s | |
| inter | 2 | 3 | $N_2 = \frac{N}{2}$ | $\frac{N^2}{4}$ |
| inter & intra | 1 | 2 | any distribution | $\frac{N(N+1)}{2}$ |
| inter & supra | 3 | no max. | $\forall l : N_l = \frac{N}{L}$ | $\frac{N^2(L-1)}{2L}$ |
| plenary | 1 | no max. | any distribution | $\frac{N(N+1)}{2}$ |

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