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On Variations of the Convex Hull Operator

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Abstract. Given a collection $\mathcal{F}$ of subsets of $I\!R^n$, consider the operator $\mathrm{hull}_{\mathcal{F}}$ associating to a set $X \subset I\!R^n$ the intersection of all elements of $\mathcal{F}$ containing X. The aim of this note is the study of the operator $\mathrm{hull}_{\mathcal{F}}$ and especially its relationship with the *convex hull* operator in the special case when $\mathcal{F}$ is the set of all half-spaces of $I\!R^n$.

1 Definitions, notations and basic properties

Definition 1.1 Given a collection $\mathcal{F}$ of subsets of $\mathbb{R}^n$ and a set $X \subset \mathbb{R}^n$, let $\mathrm{hull}_{\mathcal{F}}(X)$ denote the so called *hull of X based on $\mathcal{F}$*, defined as the intersection of all the elements of $\mathcal{F}$ containing X :

$$\mathrm{hull}_{\mathcal{F}}(X) = \bigcap_{X \subset F \in \mathcal{F}} F .$$

In this note we will also use the following notations :

- $\mathcal{P}(S)$ denotes the collection of all subsets of a given set S.
- For $X \subset \mathbb{R}^n$, X° denotes the open part of X according to the usual topology of $\mathbb{R}^n$.
- For $X \subset \mathbb{R}^n$, $\overline{X}$ denotes the closure of X according to the usual topology of $\mathbb{R}^n$.
- For any operator op : $\mathcal{P}(\mathbb{R}^n) \to \mathcal{P}(\mathbb{R}^n)$, $(\mathrm{op}(X))^\circ$ and $\overline{(\mathrm{op}(X))}$ will be denoted $\mathrm{op}^\circ(X)$ and $\overline{\mathrm{op}}(X)$ respectively.
- For any two operators op, op' : $\mathcal{P}(\mathbb{R}^n) \to \mathcal{P}(\mathbb{R}^n)$ and for any binary relation $\mathcal{R}$ over $\mathcal{P}(\mathbb{R}^n)$, we will write in short op $\mathcal{R}$ op' whenever op(X) $\mathcal{R}$ op'(X) holds for any $X \subset \mathbb{R}^n$.
- The following collections of subsets of $\mathbb{R}^n$ are associated to a collection $\mathcal{X} \subset \mathcal{P}(\mathbb{R}^n)$:

$$\begin{aligned} \mathcal{X}^\circ &= \{X^\circ : X \in \mathcal{X}\}, \\ \overline{\mathcal{X}} &= \{\overline{X} : X \in \mathcal{X}\}, \\ \widetilde{\mathcal{X}} &= \mathcal{X}^\circ \cup \overline{\mathcal{X}}. \end{aligned}$$

- $\mathcal{H}$ denotes the set of all <u>open</u> half-spaces of $\mathbb{R}^n$.
- For a point $\boldsymbol{x} \in \mathbb{R}^n$ and scalar $\epsilon > 0$, $B(\boldsymbol{x}, \epsilon)$ represents the ball —with the $\mathcal{L}_2$ norm— centered on $\boldsymbol{x}$ and of radius ϵ.

All over this note, we will make an intensive usage of some basic properties of topological spaces. In order to reference to them easily, let us first recall some of them.

Properties 1

(i) For any set A of a topological space, $A^\circ \subset A \subset \overline{A}$.

(ii) An arbitrary intersection of closed sets is closed.

(iii) An intersection of finitely many open sets is open.

(iv) For any sets A and B of a topological space, $A \subset B$ implies $A^\circ \subset B^\circ$ as well as $\overline{A} \subset \overline{B}$.

(v) For any set A of a topological space, $\overline{A}^\circ = A^\circ$.

Let us also recall some standard definitions in convex analysis. The point resulting from a weighted sum $\sum_{i=1}^k w_i \boldsymbol{x}^i$ of k points $\boldsymbol{x}^1, \ldots, \boldsymbol{x}^k \in \mathbb{R}^n$ with scalar weights $w_1, \ldots, w_k \in \mathbb{R}$ is called a *linear combination* of the points $\boldsymbol{x}^i$. Such a linear combination, with the additional property that $\sum_{i=1}^k w_i = 1$, is an *affine combination* of the $\boldsymbol{x}^i$s. If in addition, all the weights are non-negative ($w_i \geq 0$) then the affine combination of the points $\boldsymbol{x}^i$ is a *convex combination*.

The set of all linear combinations of the points $\boldsymbol{x}^1, \ldots \boldsymbol{x}^k \in \mathbb{R}^n$ is the *linear subspace spanned by the $\boldsymbol{x}^i$s*. Similarly, the set of all affine combinations of the points $\boldsymbol{x}^1, \ldots \boldsymbol{x}^k \in \mathbb{R}^n$ is the *affine subspace spanned by the $\boldsymbol{x}^i$s*. A set $X \subset \mathbb{R}^n$ is *convex* if any $\boldsymbol{x} \in X$ can be expressed as a convex combination of finitely many points $\boldsymbol{x}^1, \ldots, \boldsymbol{x}^k \in X$.

Definition 1.2 The *convex hull* of a set $X \subset I\!\!R^n$ is the set of all convex combinations of points in X. It is the smallest convex set containing X.

To conclude this first section, we will enumerate some properties of the operator $\mathrm{hull}_{\mathcal{F}}$ which will be given without demonstration because they are obvious either from Definition 1.1 or from Properties 1.

Properties 2

(i) $\mathcal{F} \subset \mathcal{G}$ implies $\mathrm{hull}_{\mathcal{F}} \supset \mathrm{hull}_{\mathcal{G}}$ for any $\mathcal{F}, \mathcal{G} \subset \mathcal{P}(I\!\!R^n)$.

(ii) $\mathrm{hull}_{\overline{\mathcal{F}}}(X)$ is closed for any $\mathcal{F} \subset \mathcal{P}(I\!\!R^n)$ and any $X \subset I\!\!R^n$.

(iii) $\mathrm{hull}_{\mathcal{F}^\circ}(X)$ is open for any finite collection $\mathcal{F} \subset \mathcal{P}(I\!\!R^n)$ and any $X \subset I\!\!R^n$.

(iv) If $\mathcal{F}$ is a collection of convex sets of $I\!\!R^n$, $\mathrm{hull}_{\mathcal{F}}(X)$ is convex for any $X \subset I\!\!R^n$.

In section 3 we will investigate the relationship between the convex hull operator and $\mathrm{hull}_{\mathcal{H}}$. Some preliminary results will be established in section 2 on relations between operators $\mathrm{hull}_{\mathcal{H}}$, $\mathrm{hull}_{\overline{\mathcal{H}}}$ and $\mathrm{hull}_{\widetilde{\mathcal{H}}}$.

2 Hulls based on open or closed half-spaces

Before considering the relation between the convex hull of an arbitrary set $X \subset I\!\!R^n$ and the intersection of all half-spaces containing X, it is necessary to clarify the situation between the various types of intersections of half-spaces that can be considered. Theorem 1 presented a general picture of the situation.

Theorem 1 Hull operators placed into a same dotted line in the following picture are equivalent :

$$\begin{tikzcd}
\mathrm{hull}^\circ_{\mathcal{H}} \arrow[r] & \mathrm{hull}_{\mathcal{H}} \arrow[r] & \overline{\mathrm{hull}_{\mathcal{H}}} \\
\mathrm{hull}^\circ_{\widetilde{\mathcal{H}}} \arrow[u] \arrow[d] \arrow[r] & \mathrm{hull}_{\widetilde{\mathcal{H}}} \arrow[u] \arrow[d] \arrow[r] & \overline{\mathrm{hull}_{\widetilde{\mathcal{H}}}} \arrow[u] \arrow[d] \\
\mathrm{hull}^\circ_{\overline{\mathcal{H}}} \arrow[r] & \mathrm{hull}_{\overline{\mathcal{H}}} \arrow[r, leftrightarrow] & \overline{\mathrm{hull}_{\overline{\mathcal{H}}}}
\end{tikzcd}$$

Proof: The most obvious inclusions $A \subset B$ are shown by arrows $A \longrightarrow B$ in the picture. Inclusions represented by left–to–right arrows follow immediately from Property 1(i). The right–to–left arrow follows from Property 2(ii) and the vertical arrows follow from Property 2(i).

To complete the proof, we will first show that

$$\mathrm{hull}_{\mathcal{H}} \subset \mathrm{hull}_{\widetilde{\mathcal{H}}}$$

which settles the equivalence of these two operators. Using this result in conjunction with Property 1(iv), we get that to prove the equivalence of the 3 operators in the left class, it is sufficient to show that

$$\mathrm{hull}^{\circ}_{\overline{\mathcal{H}}} \subset \mathrm{hull}^{\circ}_{\mathcal{H}} \, .$$

In a similar way, the demonstration of

$$\mathrm{hull}_{\overline{\mathcal{H}}} \subset \overline{\mathrm{hull}_{\widetilde{\mathcal{H}}}}$$

is sufficient to establish the equivalence of the 4 operators in the right class.

$\mathbf{hull}_{\mathcal{H}} \subset \mathbf{hull}_{\widetilde{\mathcal{H}}}$:

Take $X \subset I\!R^n$ and $\boldsymbol{x} \notin \mathrm{hull}_{\widetilde{\mathcal{H}}}(X)$. Thus, there exists $H \in \widetilde{\mathcal{H}}$ such that $\boldsymbol{x} \notin H \supset X$. If H is open, $H \in \mathcal{H}$ and $\boldsymbol{x} \notin \mathrm{hull}_{\mathcal{H}}(X)$. So, let assume that H is closed and let say $H = \{\boldsymbol{z} \mid \boldsymbol{z}^\top \boldsymbol{h} \geq h_0\}$. Define ϵ as $h_0 - \boldsymbol{x}^\top \boldsymbol{h}$, which is strictly positive since $\boldsymbol{x} \notin H$. Consider the open half-space $H' = \{\boldsymbol{z} \mid \boldsymbol{z}^\top \boldsymbol{h} > h_0 - \epsilon\}$. Note that $\boldsymbol{x} \notin H'$, while $X \subset H \subset H' \in \mathcal{H}$. Consequently, $\boldsymbol{x} \notin \mathrm{hull}_{\mathcal{H}}(X)$.

$\mathbf{hull}^{\circ}_{\overline{\mathcal{H}}} \subset \mathbf{hull}^{\circ}_{\mathcal{H}}$:

Take $X \subset I\!R^n$ and $\boldsymbol{x} \in \mathrm{hull}^{\circ}_{\overline{\mathcal{H}}}(X)$. Choose $\epsilon > 0$ sufficiently small such that $B(\boldsymbol{x}, \epsilon) \subset \mathrm{hull}_{\overline{\mathcal{H}}}(X)$. Thus, for any H such that $X \subset H \in \overline{\mathcal{H}}$, we have $B(\boldsymbol{x}, \epsilon) \subset H$. This is in particular the case for those H such that $X \subset H^{\circ}$. Since $B(\boldsymbol{x}, \epsilon)$ is open, whenever it is included in H, it is in H° as well. Thus, for any H such that $X \subset H \in \mathcal{H}$, we have $B(\boldsymbol{x}, \epsilon) \subset H$. It means that $B(\boldsymbol{x}, \epsilon) \subset \mathrm{hull}_{\mathcal{H}}(X)$, or in other words $\boldsymbol{x} \in \mathrm{hull}^{\circ}_{\mathcal{H}}(X)$.

$\mathbf{hull}_{\overline{\mathcal{H}}} \subset \overline{\mathbf{hull}_{\widetilde{\mathcal{H}}}}$:

To prove this last statement, we will show that for any $X \subset I\!R^n$, for any $\boldsymbol{x} \in \mathrm{hull}_{\overline{\mathcal{H}}}(X) \backslash \mathrm{hull}_{\widetilde{\mathcal{H}}}(X)$ and for any $\epsilon > 0$, there exists $\boldsymbol{y} \in \mathrm{hull}_{\widetilde{\mathcal{H}}}(X)$ such that $\|\boldsymbol{x} - \boldsymbol{y}\| < \epsilon$, which will imply that $\boldsymbol{x}$ is in the closure of $\mathrm{hull}_{\widetilde{\mathcal{H}}}(X)$.

Let assume, *ab absurdo*, that there is an $\boldsymbol{x} \in \mathrm{hull}_{\overline{\mathcal{H}}}(X) \backslash \mathrm{hull}_{\widetilde{\mathcal{H}}}(X)$ and $\epsilon > 0$ such that $B(\boldsymbol{x}, \epsilon) \cap \mathrm{hull}_{\widetilde{\mathcal{H}}}(X) = \emptyset$. Since $B(\boldsymbol{x}, \epsilon)$ and $\mathrm{hull}_{\widetilde{\mathcal{H}}}(X)$ are both convex (see Property 2(iv)), there exists a separating $G \in \mathcal{H}$, *i.e.* such that $\mathrm{hull}_{\widetilde{\mathcal{H}}}(X) \subset \overline{G}$ and $B(\boldsymbol{x}, \epsilon) \subset I\!R^n \backslash G$. Thus, $\boldsymbol{x} \notin \overline{G} \supset \mathrm{hull}_{\widetilde{\mathcal{H}}}(X) \supset X$ which implies that $\boldsymbol{x} \notin \mathrm{hull}_{\overline{\mathcal{H}}}(X)$, contradicting our initial assumption on $\boldsymbol{x}$.

$\triangle$

3 Convex hull versus the hull based on half-spaces

From Theorem 1, we know that the various operator hulls based on open or closed half-spaces as well as their open parts or closures, will produce either the operator $\mathrm{hull}_{\mathcal{H}}$, or its open part, or its closure. Let us now study how these three operators compare with the classical convex hull.

Theorem 2 $\boxed{\mathrm{conv}^{\circ} = \mathrm{hull}^{\circ}_{\mathcal{H}} \subset \mathrm{conv} \subset \mathrm{hull}_{\mathcal{H}} \subset \overline{\mathrm{hull}_{\mathcal{H}}} = \overline{\mathrm{conv}}}$

Proof: The establishment of these relationships between the convex hull and the hull based on half-spaces is decomposed into 2 lemmas. Lemma 3 states that $\mathrm{conv} \subset \mathrm{hull}_{\mathcal{H}}$ and Lemma 4 concerns the inclusion $\overline{\mathrm{hull}_{\mathcal{H}}} \subset \overline{\mathrm{conv}}$. From Lemma 3 we immediately get $\mathrm{conv}^{\circ} \subset \mathrm{hull}^{\circ}_{\mathcal{H}}$ and $\overline{\mathrm{conv}} \subset \overline{\mathrm{hull}_{\mathcal{H}}}$ (Property 1(iv)). To complete the proof of Theorem 2, we have to show that $\mathrm{hull}^{\circ}_{\mathcal{H}} \subset \mathrm{conv}^{\circ}$. But this follows immediately from Lemma 4 and Properties 1(iv-v). $\triangle$

Lemma 3 $\quad \mathrm{conv} \subset \mathrm{hull}_{\mathcal{H}}$.

Proof: Take $X \subset I\!R^n$ and $\boldsymbol{x} \in \mathrm{conv}(X)$. By Definition 1.2, there exist some points $\boldsymbol{x}^1, \ldots, \boldsymbol{x}^k$ in X and some non-negative scalars $w_1, \ldots, w_k$ of sum 1, such that $\boldsymbol{x} = \sum_{i=1}^{k} w_i \boldsymbol{x}^i$. Consider an arbitrary $H \in \mathcal{H}$, $H \supset X$ and let it be given by $H = \{\boldsymbol{z} \mid \boldsymbol{h}^\top \boldsymbol{z} > h_0\}$. Since $X \subset H$, $\boldsymbol{x}^i \in H$, *i.e.* $\boldsymbol{h}^\top \boldsymbol{x}^i > h_0 \ \forall i = 1, \ldots, k$. Thus, $\boldsymbol{h}^\top \boldsymbol{x} = \sum_{i=1}^{k} w_i \boldsymbol{h}^\top \boldsymbol{x}^i > \sum_{i=1}^{k} w_i h_0 = h_0$, *i.e.* $\boldsymbol{x} \in H$. Since for any H such that $X \subset H \in \mathcal{H}$ we have $\boldsymbol{x} \in H$, we conclude that $\boldsymbol{x} \in \mathrm{hull}_{\mathcal{H}}(X)$. $\triangle$

Lemma 4 $\quad \overline{\mathrm{hull}_{\mathcal{H}}} \subset \overline{\mathrm{conv}}$.

Proof: For any $X \subset I\!R^n$, the infimum of the closed convex sets containing X exists and is unique : it is the intersection of all closed convex sets containing X and we will denote it as $\widehat{X}$. The two sets $\overline{\mathrm{hull}_{\mathcal{H}}}(X)$ and $\overline{\mathrm{conv}}(X)$ are clearly closed and convex and they contain X, so each of them contains $\widehat{X}$. To prove the lemma, we just have to show that $\overline{\mathrm{hull}_{\mathcal{H}}}(X) = \widehat{X}$. Suppose *ab absurdo* that $\widehat{X} \subsetneq \overline{\mathrm{hull}_{\mathcal{H}}}(X)$ and take $\boldsymbol{x} \in \overline{\mathrm{hull}_{\mathcal{H}}}(X) \backslash \widehat{X}$. Since $\widehat{X}$ and $\{\boldsymbol{x}\}$ are closed convex and disjoint, there is a closed half-space H such that $H \supset \widehat{X}$ and $\boldsymbol{x} \notin H$. Since H contains X we have $\boldsymbol{x} \notin \mathrm{hull}_{\overline{\mathcal{H}}}(X)$ which contradicts Theorem 1 stating that $\mathrm{hull}_{\overline{\mathcal{H}}}(X) = \overline{\mathrm{hull}_{\mathcal{H}}}(X)$. $\triangle$